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Use of Streamline Coordinates in the Numerical Solution of Compressible Flow Problems

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Iterative correction of streamline geometry is used as the basis of a numerical method for subsonic isentropic steady flow problems. Direct solution of coupled ordinary differential equations is applicable to the corresponding supersonic problem. The method is formulated in three dimensions, but for illustrative purposes is applied to a two dimensional jet involving a free surface condition. The method may be particularly economical where intuition or experience provides a good starting guess for streamline geometry.

1. INTRODUCTION

A large number of numerical methods are available for problems in fluid dynamics; the surveys authored or edited by Holt [2], Keller [5], Krause [6], and Chung [1] are representative. We consider here a streamline method which appears to be very effective for at least some classes of compressible inviscid flow problems.

The formulation of the equations of motion in terms of streamline coordinates, and associated hodograph or characteristics applications, are well known (e.g., Oswatitsch [7]). Direct use of streamline coordinates for numerical purposes has also received attention; illustrative recent examples may be found in Jameson [4] and Ishii [3]. Moreover, any method in which a stream function or a potential function is obtained can be thought of as a technique for determining streamlines, at least implicitly.

We consider here a different approach, in which the goal is an explicit determination of the streamline geometry. For computational convenience, this geometry is described in terms of the intersection coordinates of streamlines with planes perpendicular to a fixed direction (the x-axis). The equation of continuity is directly integrable, and its use in the momentum equations leads to expressions for the curvatures of the streamlines. For supersonic flow, the streamlines are obtained as solutions of a set of coupled ordinary differential equations, with initial values corresponding to upstream conditions. For subsonic flow, the problem is of boundary value type (elliptic), and an iteration process is used to repeatedly correct guessed locations. The method represents a generalization of a suggestion made by Pearson [8] for incompressible flow.

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The equations for compressible, steady, isentropic flow in three dimensions are given in Section 2. However, a two dimensional problem is chosen for illustrative application of the method, in order that stability, accuracy and speed of convergence can be economically investigated. This problem is of some interest in itself, since it deals with a subsonic jet, with one free and one constrained surface, so that the boundary conditions are of mixed type. Even with a deliberately bad initial guess for the streamline configuration, convergence with 340 mesh points was fast (20 sec of CDC 7600 time) and accuracy was very satisfactory. A number of supersonic problems (see Section 4) have also been satisfactorily run, and the streamline method appears to have advantages over such conventional methods as the method of characteristics. Transonic problems, or problems involving shocks (or viscous shear effects) have not yet been considered.

A recent practical problem, to which the present streamline method has been successfully applied, is sufficiently unusual to warrant brief mention. A jet of lithium is bombarded with a beam of deuterons, so as to produce neutrons to be used in nuclear reactor materials testing. The stability of the jet is enhanced by the use of a one-sided curved constraint (much as in Fig. 3), which induces an artificial gravity field. The problem is to compute the temperature and velocity fields inside the jet, taking account of density changes resulting from the heating (apart from this effect, the fluid is considered incompressible). The method used was to combine the present streamline technique with a second iterative process, in which the temperature field is determined from the velocity field and the absorption parameters. Although viscous stresses were not included, the entrance velocity profile was chosen so as to model boundary layer effects. As in the compressible flow problem, the method converged rapidly and appeared to give accurate results.

Any iterative technique depends, for its rapidity of convergence, both on the computational algorithm and on the closeness of the initial guess to the final solution. In this latter respect, there are situations in which the present method may be particularly effective, because of the availability of an intuitive choice for streamline locations. It may also be noted that in some cases the boundary conditions are naturally expressible in terms of streamlines (e.g., airfoil surfaces, or free streamline conditions in far field).

2. Flow Equations

We consider compressible, isentropic, steady three dimensional flow, and to simplify the algebra we consider only problems in which the streamlines are nowhere perpendicular to the x-axis, so that x may be used as a convenient base variable. Referring to Fig. 1, let a typical streamline be defined by

$$\begin{aligned} y &= f(x, \alpha, \beta), \\ z &= g(x, \alpha, \beta), \end{aligned} \tag{1}$$

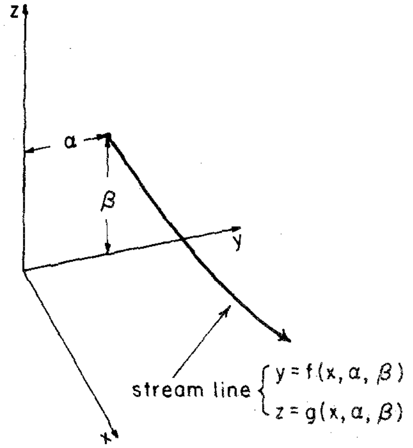

FIG. 1. Streamline geometry.

where α, β are parameters which vary from streamline to streamline, but which are fixed along any one streamline (they could, for example, denote the intersection coordinates of a streamline with the (y, z) plane, as in the figure). Denote the velocity components in the x, y, z directions by u, v, w, respectively. If $\phi(x, y, z)$ is any field variable (such as pressure), then we define the streamline derivative of ϕ by

$$\frac{d\phi}{dx} = \phi_x + \phi_y f_x + \phi_z g_x, \tag{2}$$

where a subscript denotes a partial derivative.

Let the stagnation pressure and density of the flow be p_0 and ρ_0, respectively (the same for all streamlines, for simplicity). Denote the stagnation speed of sound by c_0, where $c_0^2 = \gamma p_0/\rho_0$, γ being the ratio of specific heats. Let L be a reference length. Then writing $p = p_0 p'$, $\rho = \rho_0 \rho'$, $u = c_0 u'$, $x = Lx'$, etc., and discarding the primes, the nondimensional equations of motion become

$$(\rho u)_x + (\rho v)_y + (\rho w)_z = 0, \tag{3}$$

$$\begin{aligned} u_x u + u_y v + u_z w &= -\frac{1}{\gamma\rho} p_x, \\ v_x u + v_y v + v_z w &= -\frac{1}{\gamma\rho} p_y, \\ w_x u + w_y v + w_z w &= -\frac{1}{\gamma\rho} p_z, \end{aligned} \tag{4}$$

$$\frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{(\gamma - 1)\rho} = \frac{1}{\gamma - 1} \tag{5}$$

with $p=\rho^{\gamma}$ and $c^2=p/\rho$, where c is the local velocity of sound. We also rewrite Eq. (1) in non-dimensional form, and again drop the primes. We note that Eq. (5) implies a relationship between density and Mach number M:

$$M^2=\frac{2}{\gamma-1}\left(\frac{1}{\rho^{\gamma-1}}-1\right). \tag{6}$$

We will also make use of the relations

$$\begin{aligned} v &= uf_x, \\ w &= ug_x \end{aligned} \tag{7}$$

that follow from the definitions of f and g.

Using Eq. (2), we can now write

$$\begin{aligned} \frac{d}{dx}(\rho u) &= (\rho u)_x+(\rho u)_y f_x+(\rho u)_z g_x \\ &= -(\rho v)_y-(\rho w)_z+(\rho u)_y f_x+(\rho u)_z g_x \end{aligned}$$

by use of Eq. (3). Using Eq. (7) to replace v and w, we obtain

$$\frac{d}{dx}(\rho u)=-(\rho u)(f_x)_y-(\rho u)(g_x)_z. \tag{8}$$

Since f_x, for example, is an function of (x, α, β), we must use the chain rule to calculate $(f_x)_y$:

$$\begin{aligned} (f_x)_y &= f_{x\alpha}\alpha_y+f_{x\beta}\beta_y \\ &= (f_{x\alpha}g_\beta-f_{x\beta}g_\alpha)/J, \end{aligned}$$

where the Jacobian J is given by

$$J=f_\alpha g_\beta-f_\beta g_\alpha. \tag{9}$$

Computing $(g_x)_z$ similarly, Eq. (8) becomes

$$\begin{aligned} \frac{d}{dx}(\rho u) &= -\frac{\rho u}{J}(f_{x\alpha}g_\beta-f_{x\beta}g_\alpha-g_{x\alpha}f_\beta+g_{x\beta}f_\alpha) \\ &= -\frac{(\rho u)}{J}\cdot\frac{dJ}{dx}. \end{aligned} \tag{10}$$

Integration along the streamline yields

$$\rho u=\frac{C(\alpha,\beta)}{J},$$

where C is some function of (α, β) only. Because of the interpretation of J as an area ratio, Eq. (10) is physically clear. Using Eq. (7), we also have

$$\rho v = \frac{C}{J} f_x, \qquad \rho w = \frac{C}{J} g_x$$

so that (with $p = \rho^\gamma$) Eq. becomes

$$A = \rho^2 - \rho^{\gamma+1}, \tag{11}$$

where

$$A = \frac{\gamma - 1}{2} \cdot \frac{C^2}{J^2}(1 + f_x^2 + g_x^2). \tag{12}$$

Restricting attention for the moment to subsonic flow, suppose that a streamline configuration has been guessed, so that tentative values of f and g are known. Then, apart from the function $C(\alpha, \beta)$, all quantities occurring on the right-hand side of Eq. (12) will also be known throughout the flow field. The value of C (as a function of α and) in a typical problem can be obtained by the use of Eq. (11) at one end of the flow field (e.g., across the exit of the jet of Section 3, taken far enough downstream that the pressure is ambient), and using this value for C, the value of A will be known everywhere. Equation (11) then determines ρ at each point, and it is now necessary to investigate the extent to which ρ is in error.

Before proceeding, however, the use of Eq. (11) to determine ρ deserves some comment. A plot of this equation is given in Fig. 2, and it is clear that there is an upper limit to values of A for which a solution can be found. For ower values of A, there will be two values of ρ; one value corresponds to subsonic flow, and the other to supersonic flow. The transition point represents transonic flow. For subsonic flow, it is convenient to fit the curve of Fig. 2 with a simpler curve (with no inflection point) suitable for the early stages of the iteration; in the later stages, the exact equation (11) is used, and is solved by a Newton method.

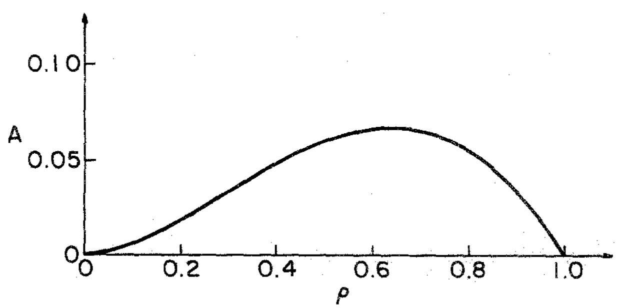

FIG. 2. Plot of A vs ρ in Eq. (11).

Returning now to the general process, suppose that Eq. (11) has been used to determine ρ at each point of the tentative flow field. Unless the streamline geometry is correct, Eqs. (4) will not be satisfied individually, although that combination of them which leads to the Bernoulli equation (5) will be satisfied because of the use of this equation to compute density via Eq. (11). It should therefore be possible to extract two independent conditions from Eqs. (4) which can be used to correct the supposed flow field as defined by the chosen values of f and g.

In the first part of Eqs. (4) use Eq. (2) to replace u_x by $(du/dx - u_y f_x - u_z g_x)$, and then use Eqs. (7) to obtain

$$u\frac{du}{dx} = -\frac{1}{\gamma\rho}p_x - \rho^{\gamma-2}\rho_x. \tag{13}$$

Thus

$$(\rho u)\frac{d}{dx}\left(\frac{\rho u}{\rho}\right) = -\rho^{\gamma-1}\rho_x$$

and, from Eq. (10), we can therefore write

$$\begin{aligned} -(\rho u)^2\frac{J_x}{J} &= \frac{(\rho u)^2}{\rho}\cdot\frac{d\rho}{dx} - \rho^\gamma\left(\frac{d\rho}{dx} - \rho_y f_x - \rho_z g_x\right) \\ &= R, \end{aligned} \tag{14}$$

say, where

$$R = \left(\frac{C^2}{\rho J^2} - \rho^\gamma\right)\frac{d\rho}{dx} + \frac{\rho^\gamma}{J}\left[(\rho_\alpha g_\beta - \rho_\beta g_\alpha)f_x + (\rho_\beta f_\alpha - \rho_\alpha f_\beta)g_x\right]. \tag{15}$$

Equation (15) gives the value of R, in terms of C, f, and g. (Conventional centered differences, using only streamline functions, can be used in Eq. (15) for the numerical process. This suggests that, to obtain second order accuracy, a uniform spacing in terms of α and β variables should be chosen for the streamlines.)

Similarly, the second and third parts of Eqs. (4) lead to

$$\begin{aligned} (\rho u)^2\left[f_{xx} - f_x\frac{J_x}{J}\right] &= (\rho u)^2\frac{f_x}{\rho}\frac{d\rho}{dx} - \frac{\rho^\gamma}{J}(\rho_\alpha g_\beta - \rho_\beta g_\alpha) \\ &= S, \end{aligned} \tag{16}$$

and

$$\begin{aligned} (\rho u)^2\left[g_{xx} - g_x\frac{J_x}{J}\right] &= (\rho u)^2\frac{g_x}{\rho}\frac{d\rho}{dx} - \frac{\rho^\gamma}{J}(f_\alpha\rho_\beta - f_\beta\rho_\alpha) \\ &= T. \end{aligned} \tag{17}$$

By combining Eqs. (15), (16), and (17), formulas for the desired curvature quantities f_{xx} and g_{xx} are obtained:

$$f_{xx} = \frac{J^2}{C^2}(S - f_x R),$$
$$g_{xx} = \frac{J^2}{C^2}(T - g_x R). \tag{18}$$

Suppose now that slope values f_x and g_x (as well as f and g, of course) are known at that point where each streamline enters the region of interest. Then Eqs. (18) may be integrated numerically so as to yield a revised streamline position, corresponding values of R, S, T again calculated, and Eq. (18) again applied. This kind of iterative process can be expected to be a sensitive one, in the sense that a slight change in streamline geometry can produce a large change in density. Consequently, iterative stabilization will be useful, and in particular it has been found that the use of a weighted combination of new and old streamline slopes is effective. It is computationally efficient to permit the weighting factor to vary during the progress of the iteration.

Supersonic flow is considered in Section 4.

3. Jet Problem

The algebra of Section 2 becomes simpler for a two dimensional problem)where streamlines are described by the single function $y = f(x, \alpha)$, and J is replaced by f_α, etc.), and for reasons of economy such a problem has been chosen for purposes of experimentation. Mixed boundary conditions have been imposed, so as to involve some degree of generality.

Consider a jet of compressible gas issuing from an orifice (at $x = 0$ in Fig. 3), and proceeding to the right. The lower surface is constrained to follow the curve shown in the figure (specifically, $y = x^2/(2\sqrt{3})$ for $0 < x < 1$, $y = (x - 0.5)/\sqrt{3}$ for $x > 1$); the upper surface is subject to ambient values of pressure, corresponding to $\rho = 0.7$. Sufficiently far downstream, the pressure across the jet should also be ambient, and we take this condition as adequately valid at $x = 3$. To make the problem well-posed, we also require a specification of flow angle across the orifice (in lieu of a specification of upstreaam geometry, to the left of the orifice), and here we take $f_x = -0.3y$ across the width $0 < y < 1$ of the orifice.

At the orifice, the streamlines are chosen to be uniformly spaced. A deliberately bad initial streamline geometry was chosen, in which the jet width at $x = 3$ was only about 40% of the correct width. The final streamlines (for $0 < x < 2$), after convergence of the iterative process, are shown in Fig. 3, and it will be seen that as a result of the turning they are compressed towards the bottom of the region. Mach number contours are also sketched, and it is found that expansion of the jet is virtually complete in that portion of it depicted in the figure.

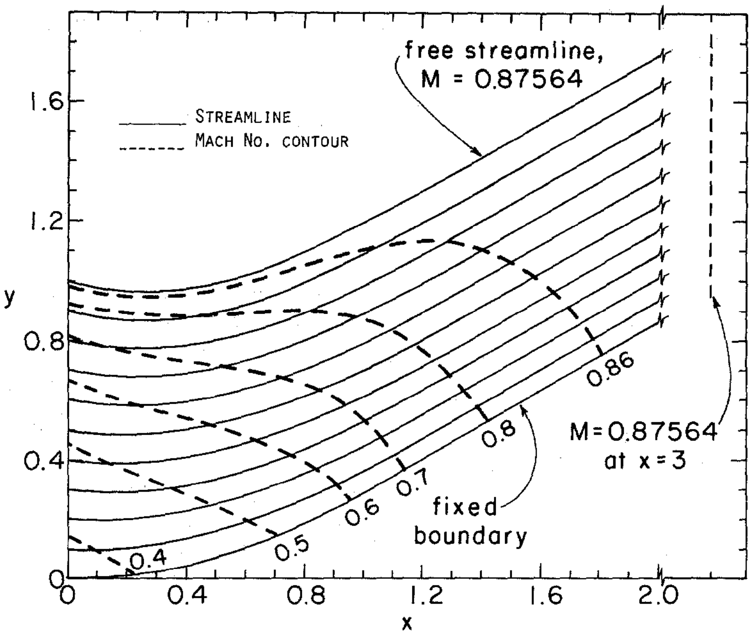

FIG. 3. Compressible jet.

A number of incompressible and compressible jet problems of this kind have been analyzed, and the accuracy of the results verified by conventional numerical tests (such as the use of a halved mesh spacing).

4. Supersonic Flow

If the iterative method of Section 3 is applied to a supersonic flow problem, the method will not converge. This result is to be expected, because the x-direction is time-like, and initial rather than boundary value data are required for a hyperbolic problem. This suggests that in a supersonic flow problem one could start by specifying conditions at some initial value of x, say at $x = 0$, and then proceed in the direction of increasing x, determining the streamline geometry so as to satisfy the equations of Section 2. If derivatives with respect to α and β are approximated by finite differences, the system reduces to a set of coupled ordinary differential equations, and conventional numerical methods are applicable.

Several two dimensional supersonic nozzle flow problems, with prescribed wall locations, have been solved by this method; the results are comparable to those obtained by the method of characteristics.

We remark finally that in some subsonic or supersonic flow cases the choice of the x-coordinate as a basic parameter is not appropriate, because of near perpendicularity to the local streamline or Mach line orientation. In such cases, one would presumably use different x-axes orientations in different regions, or alter the formulation so as to use streamline distance as the basic variable. Situations of this kind have not been investigated.

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